TOPOLOGY

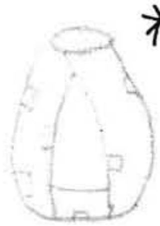
Daniel Silverman
Math 11A
April 6, 1979

HOW TO USE THIS PROJECT

This report on topology is divided into three sections. The first section is the actual reading material. The second section is the three inch column on the right hand side of each page, containing two dimensional drawings to illustrate the text. The third section is three dimensional renderings of certain two dimensional drawings. Drawings that are constructed in three dimensions are marked with asterisks.

EXAMPLE:

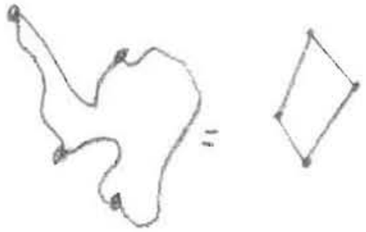
Later, mathematicians discovered a Klein bottle could be made by putting two moebius strips together.


*

After reading the text and looking at the drawing, you would refer to the Klein bottle box and look at the three dimensional construction of the Klein bottle.

TOPOLOGY

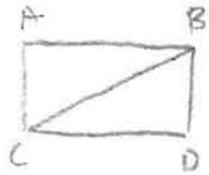
Topology is a study of lines, dots, sides and edges of geometric figures. In topology, twisting, bending, stretching and cutting of geometric figures are all permissable. For example, a square and a rectangle are topologically equivalent because both have four edges, four verticies and one hole.

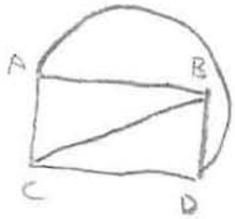


We can also use our alphabet for topological equivalencies. the letters C, I, J, L, M, N, S, V, V, W, and Z are equivalent. All have one line without any enclosed regions.

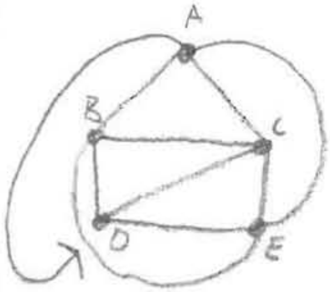
AR
B
CLMNSVVWZIJ
DO
EF
GT
HI
KX
P
Q

In topology lines are not necessarily straight. That is why a Z is considered to possess one line. So, if in regular geometry you wanted to connect each of four dots to every other dot without crossing any line, you would find that it was impossible. However, in topology this can be executed successfully.





When each dot is connected to every other dot, three lines meet at each dot. This is known as a network of degree three. If five lines met at every dot, it would be a network of degree five. But until we learn more about topology, a network of degree five is impossible.



THE MOEBIUS STRIP

* and variation on the moebius strip

A Moebius strip is simply a ring of paper with a single twist in it, but is has some very interesting qualities and properties. The moebius strip lacks orientability. Orientability is the ability to distinguish one side of a figure from the other. In other words, a moebius strip is a one-sided, one-edged piece on paper. To prove this, you will find that if you start at one point with your finger and trace it along the paper, you will eventually return to your original point.

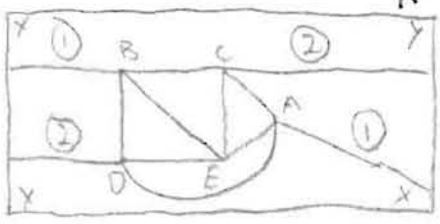
If you cut a moebius strip in half, directly down the middle, you would think that you would end up with two separate moebius strips, but in fact, you end up with one band of paper with twists, two sides and two edges.

* and $\frac{1}{2}$ inch, and $\frac{1}{4}$ inch

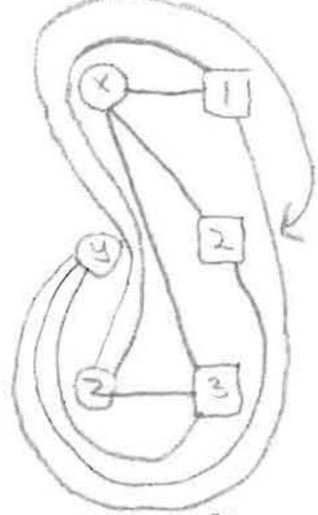
If you cut a moebius strip one third of the way from the edge you get two bands of paper and they are looped. One is a moebius strip and the other is a double sided orientable strip, twice as long as the original strip.

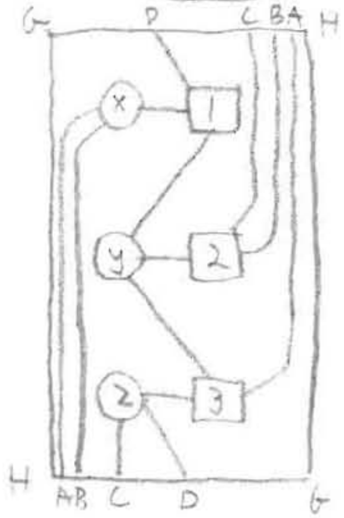
If you were to cut a moebius strip one quarter of the way from the edge, you would get two linked bands again. This time, one band would be a moebius strip with the same length as the original, and the other band is a regular band, except that it is twice as long as your original moebius strip.

A moebius strip makes some things possible that are impossible on regular paper. For example, if you wanted to connect each of five dots to every other one on regular paper, you would find that you cannot do so without crossing any lines. However, it is possible to do this on a moebius strip when you utilize its special properties.

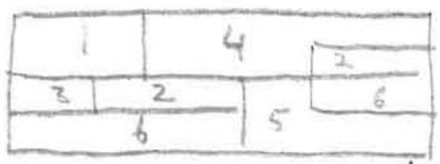


Another example of this is the following: Three houses wish to receive gas, water and electricity without having any pipes or lines crossing. You will find that this is only possible when you use the moebius strip.





When coloring a map on a flat surface, only four colors are needed. But on a moebius strip, the least number of colors needed is six.



Moebius strips are used in machinery so that the material will wear evenly, and also in tape recorders, for films, and even as an interesting plot "twist" in science fiction stories.

TOPOLOGIC EQUIVALENCIES

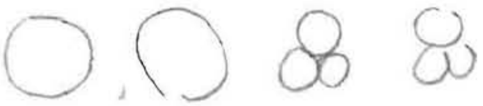
If you were to draw a square on a sheet of rubber, you could stretch the square into a plethora of different shapes. But the number of regions within the square would always remain the same -- one. When the number of regions remains the same, it is called an invariant.

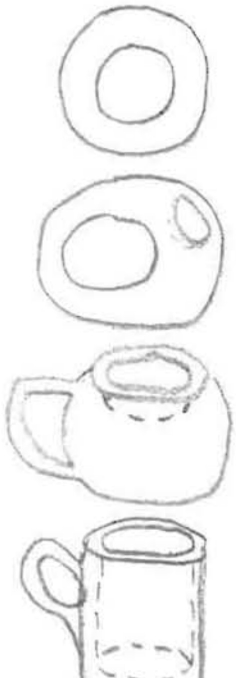
Topologists always lable an object according to the number of regions it posseses. If an object possesses no holes, it is called a genus zero. If there is one hole it is called a genus one, five holes -- genus five and so on.

genus zero

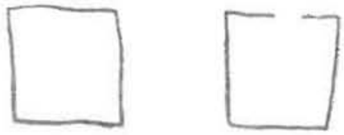
genus five

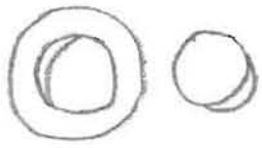
A circle must be cut once to become a genus zero. A pretzel must be cut three times to become a genus zero. There are some very unlikely shapes and objects that have the same genus number. A doughnut and a coffee cup are both of genus one. Nothing is cut or torn when converting a doughnut into a coffee cup; the material has only been reshaped.

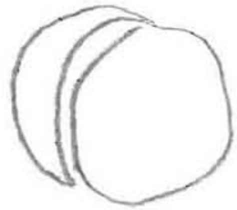




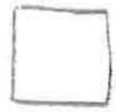
Georgio Betti was a topologist who studied methods of cutting a minimum number of cuts to attain genus zero. For example, a square would have a Betti number one, because it takes one cut to make a square genus zero. A pretzel's Betti number is three. A sphere has the Betti number zero, since it is made of one piece and has no edges. If you were to cut a hole through a sphere it would be in two pieces. If you were to cut it in half, you would end up with two half spheres.







There are some unlikely objects with the same Betti number. A piece of paper and a hollow sphere with a hole in it are topologically equivalent and have the same Betti number. Both have two sides (the sphere has an inside and an outside), one edge (the hole is the sphere's edge), and both have a Betti number of zero. The hole of the sphere in not considered a true topological hole because it does not go through the entire sphere. Therefore, the sphere and the paper are topologically equivalent.

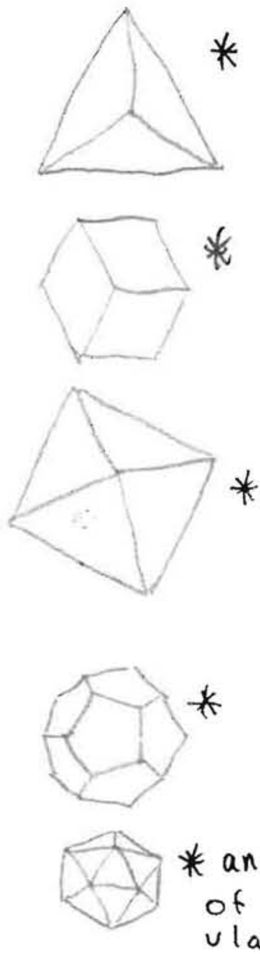
1 

2 

3 

4 

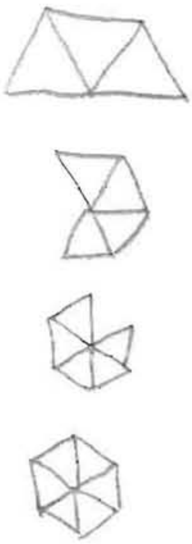
POLYHEDRONS



* and examples of non-regular polyhedrons

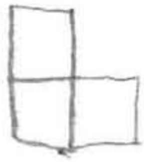
Topology is not restricted to two dimensions. Three dimensional figures can also have topological properties.

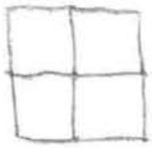
A regular polyhedron has all equal sides and equal angles. There are only five regular polyhedrons: tetrahedrons (four sides), hexahedrons (six sides), octahedrons (eight sides), dodecahedrons(twelve sides), and icosahedrons (twenty sides). There are an infinite number of non-regular polyhedrons, because there is no restriction of the shapes of the faces. The more faces however, the closer the polyhedron comes to looking like a sphere.



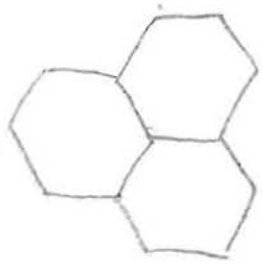
Why are there only five regular polyhedrons? Well, let's try an experiment to help answer this question. A tetrahedron has three triangles at each vertex. If we add a triangle to each vertex we get an octahedron. Add another and we get an icosahedron. But, when we add another at each vertex, the paper won't be able to fold ! So it would be impossible to add a third dimension to the figure. All we would get is a tiled plane of triangles.

Similarly, if a fourth square were added to each vertex of a cube, the paper wouldn't be able to fold itself out of the second dimension. All we would get is a tiled plane of squares.





A regular polyhedron with hexagons would not work because we would need three hexagons at each vertex, and there would be no way to fold it into the third dimension. In fact, you cannot make a regular polyhedron with more than five sides on each face.



If we explore the relationship between regular polyhedrons and the numbers of their parts;

FIGURE	FACES	VERTICES	EDGES
Tetrahedron	4	6	6
Hexahedron	6	8	12
Octahedron	8	6	12
Dodecahedron	12	20	30
Icosahedron	20	12	30

we see that there can only be 6, 12 or 30 ~~vertices~~. edges. In the hexahedron and octohedron the number of vertices and faces is reversed. The same is true for the dodecahedron and the icosahedron. Leonard Euler, a topologist, thought of the following formula for almost any two or three dimensional figure: number of vertices (V) + number of faces (F) - number of edges (E) = 2.
Let us see if his formula is applicable to regular polyhedrons.

tetrahedron	$4 + 4 - 6 = 2$
hexahedron	$6 + 8 - 12 = 2$
octahedron	$8 + 6 - 12 = 2$

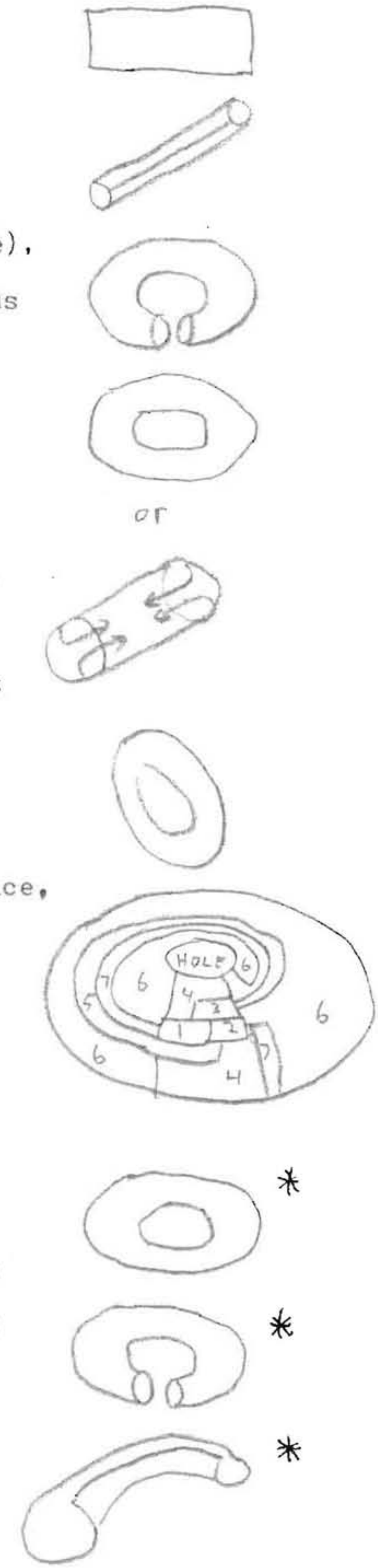
TORUSES

A torus is nothing more than a hollow doughnut. It has two sides (inside and outside), but no edges. It has one hole, so it is of genus one.

There are two ways to make a torus. The first is to take a piece of paper, roll it into a cyliner, and connect the two ends. The other, more difficult way to make a torus, is to roll a piece of paper into a cylinder and then connect the ends through the cyliner itself.

As was said earlier, a minimum of four colors are needed to color a map on a flat surface, and six colors are needed on the moebius strip. How many colors are needed to color a map on a torus? Since all colors must wrap around the entire surface, one can assume that the answer is more than six. In fact, it is seven.

The Betti number of a torus is two -- one cut to turn it back into a cylinder, and one lengthwise cut to make it a flat piece of paper again.

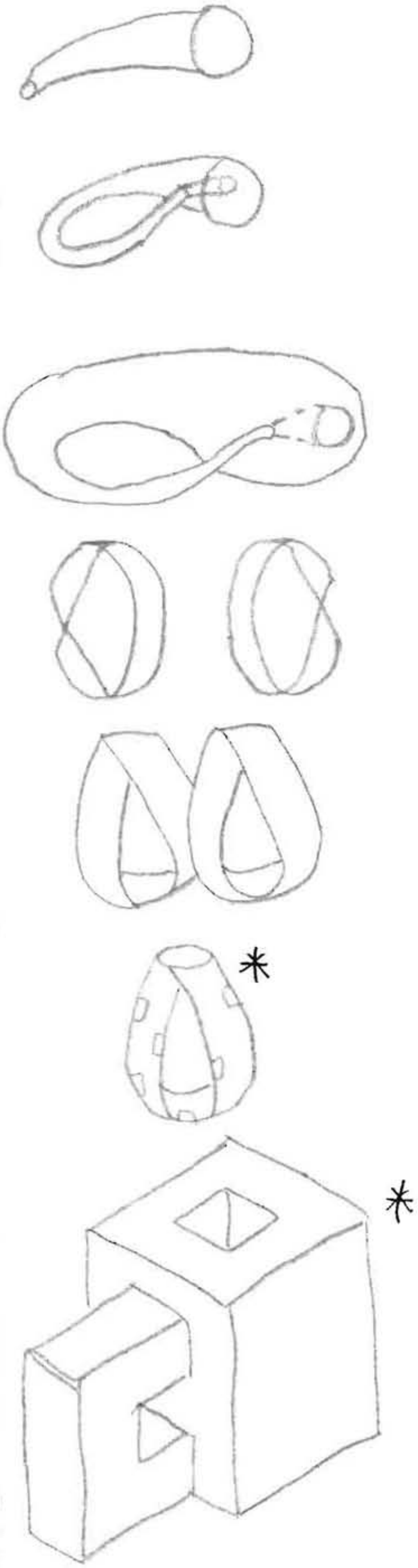


THE KLEIN BOTTLE

A German matnemetician, Felix Klein, wanted to make a bottle whose inside was on its outside. Of course, a "bottle" like this would only be imaginary, but he devised a model of a "Klein bottle" that could be constructed in reality.

He thought,"If I take a horn-shaped object, cut a hole in the horn and slip the smaller end through the hole and out the other end, and then cemented the edge of the inner hole to the edge of the outer hole, I would have a model of my bottle!"

Later, mathematicians discovered a Klein bottle could be made by putting two moebius strips together. By taking two moebius strips that are mirror images of each other, folding each at one point and taping them together except at the fold, you will have a Klein bottle



This project has only touched the surface of topology. there is much more that can be explored. working with the more abstract and complex fields of topology requires a wide knowledge of algebra and geometry. some mathematicians believe that topology will lead to yet undiscovered fields of math and science.